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INSTABILITIES OF ROTATING COMPACT STARS: A BRIEF OVERVIEW

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Abstract. Direct observations of gravitational waves will open in the near future new windows on the Universe. Among the expected sources, instabilities of rotating compact astrophysical objects are waited to be detected with some impatience as this will sign the birth of “gravitational waves asteroseismology”, a crucial way to improve our knowledge of matter equation of state in conditions that cannot be reproduced in a lab. However, the theoretical work needed to really get informations from to-be-detected signals is still quite large, numerical simulations having become a necessary key ingredient. This article tries to provide a short overview of the main physical topics involved in this field (general relativity, gravitational waves, instabilities of rotating fluids, *etc.*), concluding with a brief description of the work that was done in Paris-Meudon Observatory by Silvano Bonazzola and collaborators.

1 Introduction

Among the greatest discoveries made in astrophysics during the last decades, quasars, active galactic nuclei, gamma ray bursts and pulsars are good illustrations of the fact that general relativity (GR) is getting a growing importance in explaining physical mechanisms behind high energy astronomical observations. More precisely, those four classes of phenomena involve so-called “compact objects”, whose main properties can only be understood by taking into account Einstein’s theory of gravitation. With numerous high energy satellites (XMM-Newton, Integral, Chandra, *etc.*) now fully operative, eagerness to apprehend inner structure and dynamics of these objects or of matter surrounding them (accretion disks) has been an incitement to deepen the study of general relativistic hydrodynamics. However, another reason why the dynamics of fluids in strong gravitational fields¹ has naturally become one of the hottest topics in theoretical astrophysics is that

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¹i.e. curved spacetimes whose curvature is possibly created by the fluid itself.

some gravitational wave (GW) detectors (VIRGO, LIGO, *etc.*) are entering into the game. As this article will illustrate, material compact objects are indeed very good candidate sources of GWs, the study of their hydrodynamics being a key to understand how the emission takes place.

The present text is the written transcription of a talk I had the pleasure to give at Cargèse (Corsica, France) in May 2005. This was during a School on “Astrophysical fluid dynamics” organized by Bérangère Dubrulle and Michel Rieutord in honour of Jean-Paul Zahn and Silvano Bonazzola. The latter has been working in Paris-Meudon Observatory since 1972, his main topics of interest being relativistic astrophysics, numerical relativity and gravitational waves. In those fields, Silvano is mainly known for having initiated the use of “spectral methods”, making the Meudon group known for the high level of precision reached in its numerical works (see for instance Bonazzola *et al.* 1999). In fact, most of the people working in that group have had the preparation of their PhD thesis supervised by him², as it was my case from 1999 to 2002. The subject of my thesis was the study of some gravitational-wave driven instabilities of rotating neutron stars (NSs), the so-called r-modes, and I was kindly invited by the organizers of the Cargèse school to give a lecture on that issue. However, this subject and my common work with Silvano are profitably replaced in the more general context of instabilities of rotating compact stars, a field in which Silvano had already been involved previously with some other collaborators from Meudon (see Section 5.3). Hence, in the following, I do not pretend to be exhaustive and precise, but aim at giving a very short and general survey of some chosen studies/problems related to the question of hydrodynamical instabilities of rotating compact stars, a subject in which the emission of gravitational waves is an essential phenomenon. Yet, as the school addressed not only to astrophysicists but also to people working in hydrodynamics without any link to astrophysics, brief introductions on compact stars and gravitational waves begin this article, before it deals with oscillations and instabilities of rotating compact stars, the astrophysical relevance of the described mechanisms being discussed in the Conclusion.

2 Compact stars

Formation, structure and evolution of compact stars are complex subjects described in detail within various monographs and review articles such as Shapiro & Teukolsky (1983), Bethe (1990), Pons *et al.* (1999) and Weber (1999). Here, we shall only give a very brief summary, mainly explaining what is a “compact star” and what are its typical features.

To start with, it is maybe worth reminding that the more massive a star is at

²for instance E. Gourgoulhon, whose contribution in this Volume gives a nice introduction to relativistic hydrodynamics using Cartan external calculus, as was developed by B. Carter.

birth, the faster it evolves, producing heavier and heavier nuclei by successively making fusion of lighter ones. The most stable element being Fe^{56} , no star can use it to produce thermal energy to counterbalance the gravitational attraction. As a consequence, when a star is sufficiently massive to produce Fe^{56} nuclei, the latter begin to accumulate in its core. Being sterile, this iron core owns its survival mainly to the degeneracy of the electrons that makes the matter neutral. However, as was shown by Chandrasekhar, a self-gravitating quantum gas of fermions admits a maximal mass called the “Chandrasekhar mass” (around 1.5 Solar masses, the exact value depending on the lepton fraction). If the gas has a mass larger than this threshold value, the central fermions are so energetic that they become relativistic, which lowers the compression modulus. This generates an instability of the iron core, in such a way that when its mass reaches the Chandrasekhar value, the inner part of the star suddenly collapses³. Due to the repulsive nature of the strong interaction at small distances, the fall of the matter ends with a bounce off of the inner part when the density reaches values around twice the atomic nuclear density, also called the “saturation density” ($n_s \sim 2.5 \times 10^{14} \text{ g cm}^{-3}$). Since the outer part is still falling, a shock-wave is generated that was first thought to be responsible for the ejection of the external layers with an intense electromagnetic emission leading to a type Ib, Ic or II supernova. However, it is now known that the way toward successful supernova is more complicate, involving also for instance neutrinos and convection, the full mechanism being still unperfectly understood. Yet, the supernova being successful or not, the central remnant left, a compact warm plasma mainly composed of neutrons, protons, electrons and neutrinos called a “proto-neutron star” very fast begins to contract, while losing energy and neutrinos. Depending on the amount of matter falling back on it and on the detail of the dynamics, the proto-neutron stars gives birth to a black hole (BH), a cold neutron star or a cold strange star (SS), the latter being formed if the baryonic matter undergoes a phase transition to a quark-gluon plasma.

These classes of astrophysical objects are called “compact” due to the fact that the ratio between their Schwarzschild radius⁴ and their radius, called “compactness parameter”, is much larger than for usual objects. For instance, a typical NS has a mass similar to that of the Sun ($M_\odot \sim 2 \cdot 10^{30} \text{ kg}$), but a radius which is 10^5 smaller (around 10 km for a NS and 10^6 km for the Sun). Its compactness is thus much larger, being around 0.2 instead of 10^{-5} for the Sun. It is in fact easy to see that the compactness of any object is by definition smaller than 1, reaching values close to it only for NSs (or SSs), and being equal to it only for BHs. Incidentally, it means that the compactness of a star is a measurement of “how relativistic it is”. The closest it is to 1, the more general relativity is needed to properly describe the star, its behaviour and what happens around it. In the following, we shall deal only with the hydrodynamics inside compact stars, *i.e.*, we shall not consider

³Notice that the collapse also results from the disappearing of some energy due to electron captures and photodesintegration of iron nuclei taking place at high density.

⁴which are proportional to their masses: $R_s \equiv 2 M G/c^2$

“pure geometrical” black holes, in which, in the spherically symmetric case, all the matter is led to a central singularity. This restriction has two main consequences on the hypothesis that can be made in the description of our fluid ball:

- it is compact, composed of degenerate nucleons or quarks, gravitation being mainly counterbalanced by the strong interaction/Quantum ChromoDynamics. Due to the saturation property of the latter, it can be shown that taking a constant density profile at null temperature is a reasonable approximation;
- (almost-)conservation of angular momentum during the collapse can lead to very high angular velocities, the star possibly reaching the Kepler angular velocity for which the gravitational force at the equator is hardly equal to the centrifugal force, meaning that any additional acceleration would imply a loss of matter.

As we shall see now, these two characteristics of compact stars make them favourable for the emission of gravitational waves.

3 General relativity and gravitational waves

Introductions to the physics of gravitational waves and of their astrophysical sources can be found in the famous monograph *Gravitation* by Misner, Thorne & Wheeler (1973), or in the proceedings Deruelle & Piran (1983), Marck & Lasota (1997) and Borane *et al.* (2000). Here again, we shall only give a brief and general summary.

3.1 Stepping stones in GW history

As soon as 1916, Einstein understood that his new theory of gravitation, general relativity, could admit solutions describing ondulatory perturbations of the gravitational field that propagate at the maximal allowed velocity, c . This was the first mathematical description of “gravitational waves”, even if as early as 1907, Poincaré had already mentioned the probable existence of some “ondes gravifiques” in any relativistic theory of gravitation. In 1918, Einstein came with a formula giving the gravitational emissivity of a relativistic fluid ball with weak gravitational field and slow internal motions. As this “quadrupole formula” and the solution describing gravitational waves had been reached in a given system of coordinates and after linearizing the equations of general relativity, the question arose of the physical relevance of gravitational waves. Since GR tells us that all systems of coordinates are equivalent and none of them directly has a physical meaning, it could be thought that those waves were just “coordinate waves” without any physical content. That issue started to find an answer only when Pirani (1956) asked the physical question “What would happen to my GW detector if a GW goes through my lab?” and gave an answer to it, proving that this phenomenon would actually be linked with an energy deposit. However, the symmetric question (“what does

happen to an object emitting GWs?”) was harder to deal with, and it is only in the early eighties that it was closed with the demonstration that a gravitational system emitting GWs does really lose energy until its total energy reaches a finite positive value. This was done through various works starting with (e.g.) Bonnor (1959), Bondi *et al.* (1962) and ending with (e.g.) Schoen & Yau (1979), Ludvigsen & Vickers (1981), Witten (1981) and Schoen & Yau (1982). Nevertheless, even before the existence of GWs inside Einstein theory was theoretically proved, people had started to experimentally look for them, following a pioneer called Weber.

The story of GW detection has indeed its roots in the late sixties with the building by J. Weber of the first resonant gravitational detector (a “bar”), which was some huge and massive bulk whose eigenmodes should be excited by a coming GW. Shortly after having begun to take data, Weber effectively announced that he had successfully detected a signal. This detection was negated later, yet Weber’s work and supposed results had definitely boosted the race for GW detection, various similar bars being built around the world. As an example, the Paris-Meudon Observatory experiment can be mentioned, which was under the responsibility of Silvano and collaborators (see Fig. 1) until the end of 1974 when the experiment was given up.



Fig. 1. From left to right, Jean Thierry-Mieg, Georges Herpe, Silvano Bonazzola and Michel Chevreton in front of the Meudon GW detector.

As will be schematically explained in the next section, the first evidence of the existence of GWs could only come from astrophysics as no terrestrial object can be a relevant source. But quite ironically, that first experimental proof came almost exactly at that moment when most bar experiments were being left, this proof being based on the discovery by Taylor & Hulse in 1975 of a pulsar in a

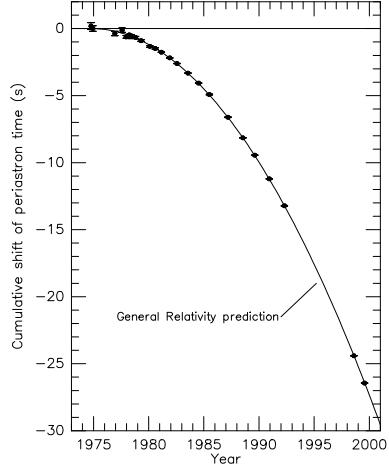


Fig. 2. Decreasing of the orbital period of the binary pulsar PSR B1913+16 measured by the cumulated shift in the time at periastron. Straight line curve is the theoretical prediction of GR [From Lorimer (2001)].

binary system (PSR B1913+16). Indeed, since it is in strong gravitational interaction, very precise measurement of the evolution of this pulsar could prove that it was behaving exactly as Einstein theory had predicted, that is to say with an acceleration of its orbital motion resulting from the loss of energy emitted as GWs (see Fig 2).

At the beginning of 2006, in spite of the discoveries of some systems with much stronger gravitational field, leading to some better tests of GR, still no direct detection of GWs has been done. Hence, this is obviously one of the main goals of the recent interferometric GW detectors (VIRGO, LIGO, LISA, *etc.*), whose main advantage with respect to bar detectors is to be sensitive to a large range of frequencies⁵. Nevertheless, it is worth reminding that even if the first direct detection of a GW signal will probably be done by one or several of those experiments (now taking data or in commissioning), they are above all gravitational telescopes and not detectors to prove the existence of GWs.

3.2 Emission of GWs in GR

The only way to really predict the GW signal emitted by a body is to solve Einstein equations with, as a source term, the energy-momentum tensor of the body and to look at what reaches infinity. However, in most of the physical situations, this can not be done analytically (mainly due to the strong non-linearities of the

⁵It should be mentioned that there are also various bar detectors, much more advanced than Weber's, making with the interferometers a whole international network of GW detectors.

theory), and in fact even using numerics, this is not such an easy task. Yet, it is not needed to exactly solve the equations to be able to find the main properties of the oscillatory solutions, as is well-known in the case of electromagnetic waves (EWs) in Maxwell theory of electromagnetism (EM). Furthermore, as we shall illustrate, looking for the differences and similarities between GR and EM turns out to be very instructive to understand the basic features of GWs and of their emission.

Electromagnetic waves can only be emitted if a given distribution of electric charges is evolving in time with a breaking of the spherical symmetry. This is related to the vectorial nature of the EM field (the photon has a spin 1) and to Gauss theorem, which exactly states that if the spherical symmetry is not broken, the electric field is constant (\equiv there is no physical scalar part in the EM field). Another formulation of that statement is that time variations of the “electric multipoles” higher or equal to the dipole are necessary. In a very similar way, it can be shown that the second order tensorial field that describes the gravitational field in Einstein theory does not include any vectorial or scalar part that would be physical. It means that the graviton is a massless spin 2 particle, but also that to emit GW, a mass-energy distribution needs to evolve in time with breaking of spherical symmetry but not only of it: axial symmetry can be preserved if and only if the radial distribution is not time invariant. Another equivalent assertion, which is by far the most precise, is that while a charge distribution needs at least time evolving dipoles to emit EWs, a mass distribution needs evolving “mass multipoles” at least of the order of the quadrupole to emit GWs.

However, we can get a better intuition on GR and GWs by going farther in this comparison with EM. Indeed, it is well-known that in Maxwell theory, a non-spherical time evolution of a charge distribution is not the only way to emit EWs: this can also be done through time variations of “magnetic multipoles”, and not only of electric ones. Since magnetic multipoles are equivalent to moving electric multipoles, one can expect from this analogy between EM and GR that a mass distribution keeping the same shape but having (at least) quadrupolar internal motions would also emit GWs through “current multipoles” or “gravitomagnetic multipoles”. An easy illustration is given by Fig. 3, which depicts a spherical ball of fluid with quadrupolar internal motions⁶. We shall not give here any more detail/justification and send the reader to a presentation of the “multipoles expansion” developed in the framework of “pseudo-Newtonian approaches to GR” in Thorne (1980) or Blanchet (2002).

Nevertheless, having encountered geometrical criteria necessary for getting

⁶This can be related with the fact that a rotating mass does not generate the same gravitational field that a non-rotating one, which is illustrated by the so-called “frame-dragging effect” that the satellite Gravity Probe B was trying to observe close to the Earth. See <http://einstein.stanford.edu/>



Fig. 3. Quadrupolar flow ($l = m = 2$) leading to a current quadrupole and the emission of gravitational waves. From K. Lockitch PhD thesis

emission of GWs is not enough. In Nature, if one wants a source to emit in a relevant way, its emissivity has also to be taken into account. Using again post-Newtonian calculations, it can be demonstrated that the gravitational emissivity of a body (with mass M , typical radius R , typical frequency of its internal motions ω and breaking of symmetry s) is given by the formula

$$\frac{dE}{dt} \sim \frac{\mathcal{G}_N}{c^5} s^2 \omega^6 M^2 R^4, \quad (3.1)$$

where \mathcal{G}_N is Newton's constant. When one puts in this formula numbers characteristics of a human size object, *i.e.* taking mass, radius, *etc.*, to be around 1 in I.S. units, the \mathcal{G}_N/c^5 factor makes that no relevant signal can be emitted since we have $\mathcal{G}_N/c^5 \sim 3 \times 10^{-53}$. In conclusion, no GW can be emitted in a lab. However, there is a very nice and easy manipulation to do, which is just to introduce the Schwarzschild radius of the source and a velocity v typical of its internal motions such that $\omega \sim v/R$. Putting this velocity in units of the light velocity, one gets

$$\frac{dE}{dt} \sim \frac{c^5}{\mathcal{G}_N} s^2 \left(\frac{R_s}{R} \right)^2 \left(\frac{v}{c} \right)^6. \quad (3.2)$$

Hence, in “astrophysically relativistic units”, the 10^{-53} factor has been replaced by its inverse 10^{+53} , meaning that the emission of GWs can be important if the candidate source

- is a compact object (high $M/R \sim R_s/R$),
- has relativistic internal motions ($v \sim c$), which are coherent (to avoid destructive interferences between various contributions to the time variations of the multipoles).

As a consequence, binary black holes, compact binaries (neutron stars, strange stars) and isolated compact object oscillating, rotating (without axial symmetry) and/or accreting are good candidate sources. However, as far as the last cases are

concerned, one can show that (hydrodynamical) oscillations are damped (at least) by the emission of GWs, which means that to avoid very short duration signals, instabilities would be more interesting than just oscillations.

4 Instabilities of rotating compact stars

4.1 Types of instabilities

The topic of GWs from instabilities of relativistic stars has been widely reviewed recently in Andersson (2003) and we send the reader to this article for more detail. We shall here just summarize the main conclusions, starting with the fact that instabilities can be classified in two categories: dynamical and secular ones. The first ones develop in a timescale that is of the same order as the (hydro)dynamical timescale, *i.e.* the period of the rotating star. While dynamical instabilities are related only to gravitation and hydrodynamics, secular instabilities are triggered by some dissipative processes acting on much longer timescales⁷. In the case of compact objects in GR, the most important sources of dissipation are viscosity and emission of gravitational radiation. In NSs, the first one mainly results either from nucleon-nucleon scattering (dominant process at low temperature) or from beta reactions $n \leftrightarrow p + e^- + \nu$ (dominant at high temperature), some other mechanisms being also involved, for instance related with possible superfluidity of the nucleons, interactions between the fluid and the crust, *etc*. The main difference between dissipation coming from viscosity and dissipation linked with emission of GWs is that viscosity does conserve angular momentum and reduce vorticity, whereas gravitational radiation takes angular momentum away from the star but without changing the vorticity. To sum up, a viscosity driven instability leads the star from a state of energy E_0 , angular momentum L_0 and circulation⁸ ζ_0 to a state ($E < E_0, L_0, \zeta < \zeta_0$) and a GW driven instability from the same initial state to ($E' < E_0, L < L_0, \zeta_0$). As we shall see in the next section, those evolutions are made possible by the apparition of some spontaneous breaking of symmetry.

4.2 Equilibrium and instabilities of rotating self-gravitating fluids

In Section 2, it was noticed that considering compact stars as having constant density profiles was not such a bad approximation due to the saturation property of strong interaction. Yet, since it is the easiest way a star can be modelized and since it enables analytical calculations, the uniform density model had been used to describe stars much before we had very precise ideas about their inner structure (see Chandrasekhar 1969). As a consequence, many results about stability of rotating stars have been obtained in the Newtonian framework, which have become classical and which are useful to have in mind before looking at the situation of

⁷Notice than in GR, gravitation is a dissipative process through GWs emission although it is not in Newtonian theory.

⁸Circulation is the flux integral of vorticity.

relativistic stratified rotating compact stars (see Andersson 2003 for a review).

The most important of those results is probably the fact that axisymmetric configurations are not always the most natural for rotating self-gravitating fluids. Indeed, even if at low angular velocity, a uniform density fluid adopts a Mac Laurin spheroid shape, when its angular velocity overreaches some value, the spheroid can become unstable and some stable triaxial ellipsoidal solutions appear. Usually, the parameter used to characterize the configuration is not the angular velocity, but the ratio between the (rotational) kinetic energy and the gravitational energy. This parameter, called β in the literature, can be shown to be always smaller than 0.5 due to the virial theorem. The relevance of β and the preference for triaxial configurations at large angular velocities can be understood in the origin of the instabilities, which is the competition between two forms of energy: rotational and gravitational. As an example, take a star with a given angular momentum J that is supposed to be conserved. The narrower its mass distribution, the smaller its moment of inertia I and the larger its rotational energy T , since it can be written $T = J^2/(2I)$. Nonetheless, its gravitational energy K is an increasing function of the mass distribution size, meaning that T and K have opposite behaviours. Hence, the larger is the angular momentum of a star, the more its energy will be dominated by the rotational part and the more stable will be the configurations with a larger moment of inertia, implying that triaxial configurations become privileged when the angular momentum is increasing. Another way to formulate this result is to say that when the value of angular momentum is sufficiently high, there is a critical value of β such that for any β larger than that critical value, some triaxial configuration is a state of lower energy than the axisymmetric one. The same line of argumentation can be followed with circulation instead of angular momentum, leading to the conclusion that there are two types of triaxial configurations relevant for increasing angular velocity, both of them being associated with some dissipative process driving the fluid to a state of lower energy. However, before giving more detail on this, let us notice that, using some “free-energy approach”, Bertin & Radicati (1976) and Christodoulou *et al.* (1995) have shown that those transitions⁹ can be understood as some spontaneous breaking of symmetries, with for order parameter one minus the ellipticity of the star, *i.e.* $1 - x$ where x is the ratio between the lengths of the two axis orthogonal to the rotation axis.

As reminded in Section 4.1, viscosity conserves angular momentum while dissipating vorticity/circulation. Hence, an unstable axisymmetric star under the influence of viscosity will be led to the first class of ellipsoids called Jacobi ellipsoids, which are rigidly rotating configurations with same angular momentum as the initial configuration but smaller energy and without circulation. As these configurations are triaxial and rigidly rotating, they do have time varying mass quadrupoles and can emit GWs, while viscosity no longer influences them. The

⁹and also a dynamical instability which occurs in the case of inviscid fluids at larger β

second class is called Dedekind ellipsoids and is the final state of unstable axisymmetric stars led to instability by GWs emission. Those ellipsoids are characterized by their apparent immobility (in the inertial frame) and constant vorticity. Their triaxial shape is supported by some internal fluxes and differential rotation. As a consequence, they do not emit GWs, but can lose energy due to viscosity.

The possible appearance of these two types of triaxial configurations is illustrated in Fig. 4 (extracted from Andersson 2003 to which the reader is sent for more detail), where has been taken into account another important result: the fact that these two instabilities admit the same critical value of β for incompressible Newtonian stars. However, this result should not make us forget that the relevance of those instabilities in physical stars is with no doubt an issue much more complex than what was briefly presented here, at least since compact stars are relativistic and compressible. Yet, this conclusion is also supported by the work of Lindblom & Detweiler (1977) who have shown that due to the antagonistic roles of viscosity and GWs emission, the two secular instabilities of Mac Laurin spheroids discussed here tend to cancel each other, the ratio of their strengths being the key to decide which of them has the final answer. Before explaining how Silvano and collaborators tried to give some elements of answer to that question and to the question of the relevance of GW emission by rotating compact stars, we shall now try to clarify the link between those instabilities and some oscillatory modes of rotating compact stars.

5 Oscillations and instabilities

Since oscillations of self-gravitating fluids have been the subject of numerous studies and books, we shall just give here two classical references: Unno *et al.* (1979) and Kokkotas & Schmidt (1999), the second one dealing with relativistic compact objects. However, most of the topics discussed in the present article are deeper developed in Andersson (2003).

5.1 Effects of rotation on oscillatory modes of stars

To illustrate the connection between instabilities and oscillations of rotating stars, it is worth first reminding the main effects of rotation on oscillatory modes. As we are dealing with stars without any other anisotropic physics than rotation (we neglect stress in the crust, magnetic field, *etc.*), it is useful to work with spherical harmonics decomposition of the functions that describe the modes. With this approach, a mode of oscillation is characterized by two “quantum” numbers (l, m) such that $m \in [-l, l]$, its frequency being ω_{lm} . In a spherically symmetric situation (*i.e.* without any rotation), it is well-known that all modes with the same azimuthal number m do have the same frequency $\omega_{lm} \equiv \omega_l$. An important physical quantity reached from this frequency is the “pattern speed”, which depicts the apparent displacement of the wave associated to the mode with respect to the star. For positive m , we get the pattern speed $\sigma_- = -\frac{w}{m} = \frac{d\phi}{dt} < 0$, meaning

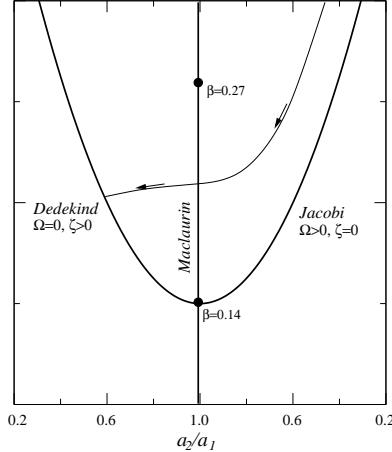


Fig. 4. Schematic illustration of the two secular instabilities discussed here as function of the β ratio (see text for definition) and the ellipticity a_1/a_2 (ratio between the axis orthogonal to the rotation axis). The critical value for both types of secular instability is around 0.14 (exactly equal for constant density Newtonian fluids), while dynamical instability is associated with a higher critical value $\beta_c \sim 0.27$. As explained in the text, Jacobi ellipsoids are favoured by viscosity, while gravitational radiation drives the fluid toward a Dedekind ellipsoid. The evolution pictured in the figure is an example of a situation in which the GW driven instability wins. From Andersson (2003)

that the mode is “retrograde”, the wave moving in the direction of decreasing ϕ . For negative m , the pattern speed is $\sigma_+ = -\frac{w}{m} = \frac{d\phi}{dt} > 0$, meaning that those modes are prograde.

When the star is rotating, all the physics is made more complicated, from the inner structure calculation (see Stergioulas 2003 for a review of rotating stars in relativity) to the behaviour of the modes which become coupled. However, even before this coupling, the main effects of rotation are the splitting and modification of the frequencies, the frequency in the inertial frame (w_i) being related to the frequency in the frame corotating with the star at angular velocity Ω (w_r) by

$$w_i = w_r - m\Omega + C_{lm}(\Omega). \quad (5.1)$$

In this formula, $C_{lm}(\Omega)$ comes from the modification of the inner structure and is thus a term of second order in Ω . Due to the $m\Omega$ term, prograde and retrograde modes are affected differently, the sign of their frequencies possibly changing, meaning a change of their apparent direction of propagation. As we shall see now, this phenomenon is an indicator of instabilities.

5.2 Oscillations and criteria for finding instabilities

Telling if a given equilibrium configuration of a self-gravitating fluid is stable or not has always been a complicate problem, the negative answer being however easier to prove. Indeed, in this case, it is sufficient to find a criteria that can indicate the existence of an instability, and not necessarily of all of them. As far as GR and GWs are concerned, an important step forward was done during the seventies by Chandrasekhar, Friedman and Schutz leading to what is now known as the CFS criteria for instability (see for instance Chandrasekhar 1970, Friedman & Schutz 1975, 1975b). What they discovered is that, in GR as in the Newtonian and post-Newtonian studies, it was possible to find non-axisymmetric instabilities of relativistic rotating stars by defining some “canonical energy” and looking for the appearance of “neutral modes” (modes with null frequencies). We shall now illustrate this with the modes associated to the two secular instabilities presented earlier and with another one discovered at the end of the nineties by Andersson (1998).

5.2.1 Instabilities and triaxial configurations

A triaxial shape is associated with $l = |m| = 2$ spherical harmonics numbers. As a consequence, it should not be a surprise that the so-called “bar-mode” ($l = |m| = 2$ f-mode) is the one that is driven to instability by viscosity or by GW emission in the case of (respectively) Jacobi or Dedekind ellipsoids appearance. Indeed, f-modes are mainly surface waves, and the $l = |m| = 2$ modes will thus correspond to surface deformations of the same shape as the ellipsoids. The only possibly missing point in this intuitive argumentation is that for the mode to imply a deformation of the Mac Laurin spheroid toward one of those ellipsoids, a synchronization between the mode and the ellipsoid to come seems to be needed for energy to transfer. In the case of the Dedekind ellipsoid, which has a fixed shape in the inertial frame, the mode needs to have a vanishing pattern speed in the inertial frame, while in the case of Jacobi ellipsoid (rigidly rotating), the mode should have a null frequency in the rotating frame. Those conclusions can be shown to be in agreement with calculations done using the canonical energy formalism developed by Friedman and Schutz, a null frequency meaning a change of sign of that energy. The resulting evolution of the bar-modes pattern speeds as functions of β (or the angular velocity) are depicted on Fig. 5 (from Andersson 2003 in which more detail can be found).

5.2.2 GW driven instabilities and multipoles

The CFS criteria very briefly introduced at the beginning of the current section has a very crucial implication that we should now discuss. Indeed, even if the frequency of any mode is modified by rotation, this shift [C_{lm} in Eq.(5.1)] will never prevent the existence of a neutral point in the inertial frame. As a consequence, for all

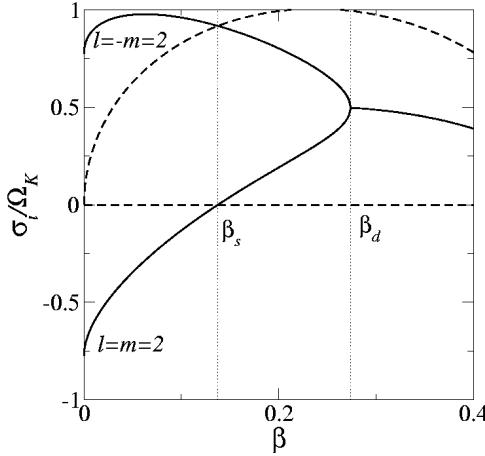


Fig. 5. Pattern speeds in the inertial frame of the $l = |m| = 2$ f-modes of a Mac Laurin spheroid as functions of the β parameter. Additionally, in dashed-line is depicted the “vanishing pattern speed in the rotating frame”. It is seen that the $m = 2$ retrograde mode is driven to instability for $\beta = \beta_s$ because its pattern speed vanishes. As explained in the text, for this neutral point is obtained in the inertial frame, it is linked with a GW driven instability leading to Dedekind ellipsoid. However, since the Mac Laurin spheroid has a constant density profile and is Newtonian, we observe that the prograde mode is driven to instability by viscosity for the same value of β , which is observed by the crossing of its pattern speed curve with the (dashed) curve of “vanishing pattern speed in the rotating frame”. Another observation that can be made albeit we shall not discuss it here is the fact that the appearance of the dynamical instability ($\beta = \beta_d$) can be found by the point at which the pattern speeds in the inertial frame of the two modes coincide. Finally, notice that all frequency are normalized by the Kepler angular velocity.

modes there is a minimal angular of the star Ω_{lm} such that the mode is driven to instability by GW emission if the star rotates faster than this. Said in another way: all relativistic rotating stars are generically unstable. However, this conclusion is valid only for inviscid fluids. As already explained, viscosity makes the game more complex, and in practice, it appears that for increasing m

- the Ω_{lm} value decreases, making the instability easier to reach
- the viscosity timescale decreases, so that viscosity can kill the candidate instability faster
- the growing timescale for the instability increases, making it more difficult to observe¹⁰.

¹⁰this result comes from post-Newtonian multipoles calculation

Thus, it can be shown that only a few modes are actual candidates for GW driven instability, each of them being characterized by a “window of instability” in the (*Temperature, Angular velocity*) plane. This is illustrated in Fig. 6 with the example of the $l = m = 2$ f-mode which was quickly recognized as a good candidate due to its quite short timescale for the growing of the instability. However, as appeared in that figure, the star needs to rotate very fast (more than 90% of its Kepler velocity) for the instability to grow.

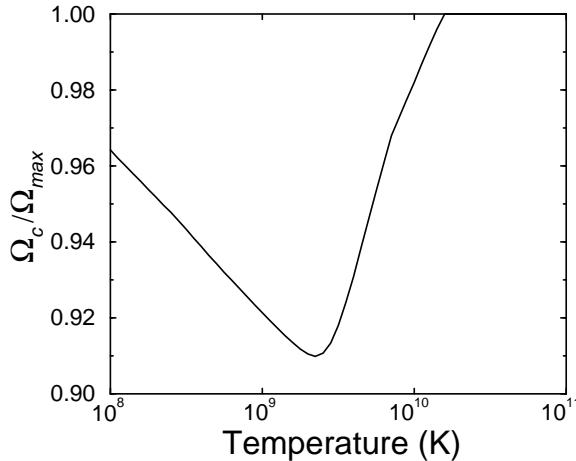


Fig. 6. Window of instability for the $l = m = 2$ f-mode with the angular velocity in units of the Kepler angular velocity (maximal angular velocity at which gravitation can balance centrifugal force at the equator). Notice that the minimal value of the angular velocity for instability is $> 90\% \Omega_K$. From Andersson (2003)

The situation slightly changed in 1998 with a discovery made by Andersson (1998) who found that even if non-axisymmetric modes without huge density perturbation fluxes have consequently small mass multipoles and can seem not very promising for the emission of GWs, they are maybe more so than expected. More precisely, he observed that some purely axial inertial modes¹¹ were unstable whatever the angular velocity of the star: they are always prograde in the inertial frame and retrograde in the rotating one, emitting GWs through current multipoles. Yet, this was just the beginning of the story of r-modes since the question of their relevance in actual stars is still open, the importance of viscosity strongly depending on the not-so-well-known equation of state and on other unknown features of NSs (see Fig. 7 to compare the size of the window of instability of the $l = m = 2$ r-mode with that of the $l = m = 2$ f-mode, which was the best candidate for

¹¹an axial mode is a mode whose spherical harmonics decomposition is such that for a given l the parity of the mode is $(-1)^{l+1}$. An inertial mode is a mode whose restoring force is Coriolis force.

GWs emission before 1998). But in a more general way, it is the whole problem of instabilities of rotating compact stars that still relies on unknowns linked with microphysics. We shall give now some words on trying-to-be realistic studies Silvano performed with various collaborators.

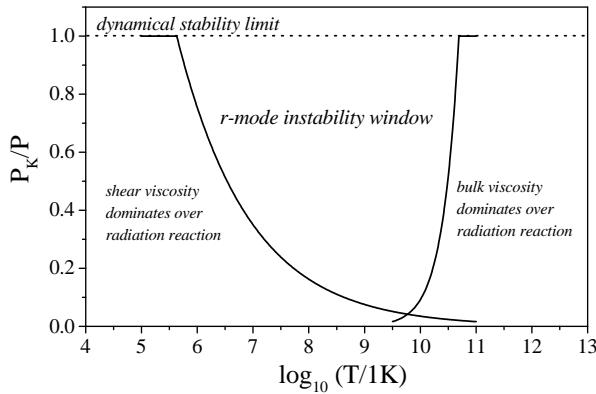


Fig. 7. Window of instability for the $l = m = 2$ r-mode. From Andersson (2003)

5.3 Some works by Silvano and collaborators

All contributions of Silvano to the study of rotating relativistic stars instabilities that will be discussed now share the use of the numerical methods he had the great idea to develop in numerical relativistic astrophysics, the already mentioned spectral methods.

5.3.1 On viscosity driven instabilities

Viscosity driven instabilities lead unstable Mac Laurin spheroids to rigidly rotating Jacobi ellipsoids. However, these ellipsoidal shapes are exact solutions only in Newtonian gravitation and with constant density profiles. Silvano and his collaborators (J. Frieben and E. Gourgoulhon) were the first to study the stability of rotating compact stars with realistic equations of states in general relativity (Bonazzola *et al.* 1996, 1998). Using a slightly modified version of a code well-known from people working in numerical relativity (Bonazzola *et al.* 1993), they proved that relativity had a stabilizing effect of self-gravitating rotating fluids with respect to the viscosity driven instability (in other words, β_c was found larger than in Newtonian theory), while also testing the precision of the code by looking for

the maximal adiabatic index of a Newtonian polytrope¹². To end with viscosity driven instabilities, it can be mentioned that a step further in the development of the numerical strategy was covered by several common collaborators of Silvano by using a multi-domain approach which enable spectral algorithms to deal with discontinuous functions (Gondek-Rosińska & Gourgoulhon 2002, Gondek-Rosińska *et al.* 2003). In this way, they were able to get more digits in the calculation of the standard problems of incompressible Newtonian fluids, but also to study the case of strange stars for which a surface density discontinuity must be taken into account.

5.3.2 On gravitational emission driven instabilities

The last subject that will be mentioned as an illustration of Silvano's achievements in the topic of rotating relativistic stars instabilities is what I had the gladness to work on in collaboration with him, mainly the instability of r-modes. The approach to that instability was different than what he did earlier due to the fact that it does not lead the star to a known state. Thus, the strategy was to develop an hydrodynamical spectral code to solve the relativistic Euler equation in realistic NSs. The first version of the code had a basic description of the microphysics (Villain & Bonazzola 2002) while we later improved it in collaboration with P. Haensel (Villain *et al.* 2005) to take into account stratification and frozen composition. It enabled us to make the first relativistic study of the coupling between gravity modes coming from stratification and r-modes. However, even if some influence on the latter was highlighted, as any work that can be done nowadays is much too imperfect, the question of the relevance of their instability from the point of view of GW emission is still quite open as we shall sum up in the conclusion.

6 Conclusion: Astrophysical relevance of the instabilities

As conservation of angular momentum says that compact stars are born with quite a fast rotation, all instabilities presented in this article (and several more) could develop in newly born NSs, unless they are generically forbidden by some phenomenon¹³. Yet, among the many unknowns concerning young compact objects, crucial issues are their profiles of rotation and their actual angular velocities at birth¹⁴. Indeed, the importance of processes like magnetic braking to slow down young compact stars is still under question, while instabilities could also be born

¹²It was indeed known that for a Newtonian polytrope, *i.e.* an equation of state for which pressure is related to density by a power law, if it is too compressible, the Kepler limit is reached before any viscosity driven instability can appear.

¹³During some time, high viscosity resulting from superfluidity of hyperons was suggested to play this role, but it seems now that some numbers had been overestimated.

¹⁴Related to these topics also can be mentioned some works based on Silvano's development of spectral methods in Meudon: Goussard, Haensel & Zdunik (1997), (1998) and Villain *et al.* (2004).

in proto-neutron stars resulting from neutron star binaries coalescences. However, the role of differential rotation is not clear either since it allows larger values of the β parameter, but also possibly creates new instabilities appearing at low β . Consequently, young compact stars are still the most natural objects from which instabilities sources of GWs can be expected, but as was proposed by Wagoner and others, those instabilities could also be influential in low mass X-rays binaries (LMXB), systems in which old compact objects continuously accrete matter from low mass star. This could be the way to a long periodic gravitational signal and from it the definitive birth of “asteroseismology of compact stars” (For more detail on these scenarios and other, see Kokkotas & Stergioulas 2005).

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